J Stat Phys (2007) 129: 193–203
DOI 10.1007/s10955-007-9397-7

One Dimensional Nearest Neighbor Exclusion Processes in Inhomogeneous and Random Environments

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Received: 21 July 2006 / Accepted: 8 January 2007 / Published online: 14 September 2007

Abstract The processes described in the title always have reversible stationary distributions. In this paper, we give sufficient conditions for the existence of, and for the nonexistence of, nonreversible stationary distributions. In the case of an i.i.d. environment, these combine to give a necessary and sufficient condition for the existence of nonreversible stationary distributions.

Keywords Exclusion processes · Nonreversible stationary distributions

1 Introduction

Transport phenomena for noninteracting particles in one dimensional environments is a well studied subject in the contexts of classical and quantum systems. However, the influence of interactions among particles in these situations is considerably less well understood. In this paper, we consider one of the more important models of particle motion with interaction—the exclusion process. In particular, we will study the exclusion process on Z^1 with nearest neighbor jumps with probabilities $p_i \in (0,1)$ and $q_i = 1 - p_i$ from i to $i+1$ and $i-1$ respectively. This is the continuous time Markov process η_t on $\{0,1\}^{Z^1}$ with formal generator

$$\mathcal{L}f(\eta) = \sum_i \{\eta(i)[1-\eta(i+1)]p_i + \eta(i+1)[1-\eta(i)]q_{i+1}\}[f(\eta_{i,i+1}) - f(\eta)],$$

where $\eta_{i,i+1}$ is obtained from η by interchanging the ith and $(i+1)$st coordinates. The exclusion process has been the object of a lot of attention over the past 35 years, primarily in

Research supported in part by NSF Grant DMS-0306167 (Chayes) and DMS-0301795 (Liggett).

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case the transition probabilities are translation invariant—see [1] and [2]. Here, as the title indicates, we investigate problems where the p_i are inhomogeneous, with particular applications to the case in which the p_i are i.i.d. random variables. Among the few rigorous papers dealing with spatially inhomogeneous asymmetric exclusion processes are [6] and [5].

The i.i.d. model was investigated in a certain approximate fashion in [11] and, more recently, in [4]. In general, these authors have studied the "phase diagram" of (maximal) current flow as a function of an equilibrium particle density parameter in the presence of disorder. Certain interesting phenomena have been uncovered in these works. In particular, a symmetric flat region in the above mentioned response curves, indicating a forbidden interval of densities, and the observation, primarily numerical, in the earlier reference that in the presence of a *random* locally preferred direction of flow, the current density vanishes with increasing system size. Our results bolster some of these conclusions: we demonstrate the existence of current carrying states whenever, for some $\epsilon > 0$, $p_0 \geq \frac{1}{2} + \epsilon$. Moreover, the flux in these states vanishes linearly with ϵ. Furthermore, we vindicate conclusively the later phenomenon. In particular, whenever $p_0 < q_0$ and $p_0 > q_0$ have positive probability—or even if $p_0 = \frac{1}{2}$ is in the support of the disorder distribution—we show that the current indeed vanishes as the system size tends to infinity. Furthermore, under the general condition of zero current, we can characterise all the invariant measures.

This picture is in sharp contrast to the non-interacting version of this problem: at the end of this paper, we will consider briefly the non-interacting case, and show that there is a stationary distribution for the system with nonzero flux whenever the p_i's are i.i.d. and $E\log(p_i/q_i)$ exists and is nonzero. This difference between the interacting and noninteracting systems might be a bit surprising, since at least at low densities, one might expect the two systems to have similar properties.

Returning to the exclusion process, we first note that reversible stationary distributions always exist in our situation. To define them, let π_i be defined by taking $\pi_0 > 0$ and then $\pi_i p_i = \pi_{i+1} q_{i+1}$ for all $i \in Z^1$. The corresponding reversible measure is the product measure ν_α, where

$$\nu_\alpha\{\eta : \eta(i) = 1\} = \alpha_i = \frac{\pi_i}{1+\pi_i}. \tag{1.1}$$

(See Sect. VIII.2 of [9], for example.) According to Theorem 2.1 of [5], these are extremal in the class $\mathcal{I}$ of all stationary distributions if and only if

$$\sum_i \alpha_i(1-\alpha_i) = \infty. \tag{1.2}$$

When this sum is finite, extremal stationary distributions μ_n (with $n \geq 0$ or $-\infty < n < \infty$, depending on the situation) are constructed from these product measures in the following way: $\mu_n(\cdot) = \nu_\alpha(\cdot \mid A_n)$, where

$$A_n = \begin{cases} \{\eta : \sum_i \eta(i) = n\} & \text{if } \sum_i \alpha_i < \infty, \\ \{\eta : \sum_i (1-\eta(i)) = n\} & \text{if } \sum_i (1-\alpha_i) < \infty, \\ \{\eta : \sum_{i\in T} \eta(i) - \sum_{i\notin T}(1-\eta(i)) = n\} & \text{otherwise,} \end{cases}$$

where in the third case, T is chosen so that $\sum_{i\in T}\alpha_i < \infty$ and $\sum_{i\notin T}(1-\alpha_i) < \infty$. These conditional measures do not depend on α, since an irreducible positive recurrent Markov chain has a unique stationary distribution. In the third case, $\{\mu_n\}$ does not depend on the choice of T, except for a possible relabelling.

In the spatially homogeneous case $p_i \equiv p$, the extremal stationary distributions are completely known [6]:

$$\mathcal{I}_e = \{\nu_\rho, 0 \le \rho \le 1\}$$

if $p = 1/2$ and

$$\mathcal{I}_e = \{\nu_\rho, 0 \le \rho \le 1\} \cup \{\mu_n, -\infty < n < \infty\}$$

if $p \neq 1/2$. Note that in the latter case, ν_ρ is not reversible. (Here ν_ρ denotes the homogeneous product measure of density ρ.)

The spatially inhomogeneous case in which Z^1 is replaced by $\{0, 1, 2, \ldots\}$ was treated in [6]. In that case, the result is that all stationary distributions are reversible.

Our main objective in this paper is to say what we can about the following question: In the spatially inhomogeneous case on Z^1, when do nonreversible stationary distributions exist? We will give sufficient conditions for the existence and for the nonexistence of such distributions; they become necessary and sufficient in the case of i.i.d. p_i's.

An important tool in discussing this issue (as in many involving the exclusion process) is the flux. For $\mu \in \mathcal{I}$, this is defined by

$$\phi(\mu) = p_i \mu\{\eta : \eta(i) = 1, \eta(i+1) = 0\} - q_{i+1}\mu\{\eta : \eta(i) = 0, \eta(i+1) = 1\}. \tag{1.3}$$

The fact that this quantity is independent of i can be checked by using $\int \mathcal{L} f\, d\mu = 0$ for $f(\eta) = \eta(i)$. In case $p_i \equiv p$, the flux under ν_ρ is $(p - q)\rho(1 - \rho)$, for example.

It is easy to check that the flux is zero for the reversible stationary distributions described above. Our first result is a converse to this observation.

Theorem 1 *Suppose $\mu \in \mathcal{I}_e$ and $\phi(\mu) = 0$. If μ is not the pointmass on $\eta \equiv 0$ or on $\eta \equiv 1$, then $\mu = \nu_\alpha$ for some α satisfying (1.1) if (1.2) is satisfied and $\mu = \mu_n$ for some n if not.*

For applications of Theorem 1, it is useful to make the following observation:

Proposition 1 *Take $\epsilon > 0$. Suppose that for each n there exists a k so that $p_i \ge \frac{1}{2} - \epsilon$ for all $k \le i \le k + n$. Then $\phi(\mu) \ge -\epsilon$ for all $\mu \in \mathcal{I}$. Similarly, if for each n there exists a k so that $p_i \le \frac{1}{2} + \epsilon$ for all $k \le i \le k + n$, then $\phi(\mu) \le \epsilon$ for all $\mu \in \mathcal{I}$.*

Theorem 1 and Proposition 1 will be proved in Sect. 2. Combining these two results, we obtain a sufficient condition for all stationary distributions to be reversible:

Corollary 1 *Suppose that for every $\epsilon > 0$ and every positive integer n there exist k and l so that $p_i \ge \frac{1}{2} - \epsilon$ for all $k \le i \le k + n$ and $p_i \le \frac{1}{2} + \epsilon$ for all $l \le i \le l + n$. Then $\mu \in \mathcal{I}$ implies that μ is reversible.*

This conclusion that all stationary distributions are reversible under the much stronger assumption that

$$\lim_{i \to \pm\infty} p_i = \frac{1}{2}$$

is a consequence of Theorem 1.1 of [5]. This conclusion also follows from Theorem 1.2 in that paper (assuming $\inf_i p_i > 0, \inf_i q_i > 0$) in all cases *other than* (a) $\lim_{i\to-\infty} \pi_i = 0$, $\lim_{i\to+\infty} \pi_i = \infty$ or (b) $\lim_{i\to-\infty} \pi_i = \infty$, $\lim_{i\to+\infty} \pi_i = 0$. When the p_i's are i.i.d., these excluded cases are of course the prevalent ones: case (a) occurs when $E\log(p_0/q_0) > 0$ and case (b) when $E\log(q_0/p_0) > 0$.

To prepare for the next result, we define the exclusion process on $[m,n] = \{m,\dots,n\}$ with boundary conditions by allowing the usual transitions in $\{m,\dots,n\}$ together with the transitions $0 \to 1$ at site m at rate p_{m-1} and $1 \to 0$ at site n at rate p_n. This is a finite state irreducible Markov chain, and hence has a unique stationary distribution $\mu_{m,n}$.

Theorem 2 *The flux $\phi(\mu_{m,n})$ is an increasing function of $p_{m-1}, p_m, \dots, p_n$.*

Combining this with the known behavior of $\mu_{m,n}$ in the homogeneous case yields the following sufficient condition for the existence of a nonreversible stationary distribution:

Corollary 2 *If for some $\epsilon > 0$, $p_i \ge \frac{1}{2} + \epsilon$ for all i, then there exists a (nonreversible) $\mu \in \mathcal{I}$ with $\phi(\mu) \ge \frac{\epsilon}{2}$.*

This answers a question raised near the end of the introduction to [5]. Theorem 2 and Corollary 2 will be proved in Sect. 3.

Combining Corollaries 1 and 2, we have the following result for the exclusion process in which the p_i's are chosen randomly in an i.i.d. fashion:

Theorem 3 *Consider an exclusion process with i.i.d. p_i's. The following hold with probability* 1:

(a) *If for every $\epsilon > 0$, $P(p_0 \ge \frac{1}{2} - \epsilon) > 0$ and $P(p_0 \le \frac{1}{2} + \epsilon) > 0$, then all stationary distributions are reversible.*
(b) *If for some $\epsilon > 0$, $P(p_0 \ge \frac{1}{2} + \epsilon) = 1$ or $P(p_0 \le \frac{1}{2} - \epsilon) = 1$, then there exists a nonreversible stationary distribution.*

Remark 1 Part (b) clearly follows from Corollary 2 if we replace the i.i.d. assumption with the assumption that $\{p_i, i \in Z^1\}$ be stationary and ergodic. The same is not true for part (a). To see this, consider the case of deterministic p_i's with

$$p_i = \begin{cases} \alpha & \text{if } i \text{ is even,} \\ \beta & \text{if } i \text{ is odd.} \end{cases} \tag{1.4}$$

Then all stationary distributions are reversible if and only if $\alpha + \beta = 1$. Indeed, if $\alpha + \beta = 1$, then the process is symmetric, so all stationary distributions are exchangeable (and therefore reversible in this case) by Theorem 1.12 of Chap. VIII of [9]. On the other hand, the homogeneous product measures ν_ρ are stationary for all choices of α and β by Theorem 2.1(a) of the same chapter. However, since

$$\phi(\nu_\rho) = (\alpha + \beta - 1)\rho(1-\rho),$$

ν_ρ is not reversible if $\alpha + \beta \ne 1$. By letting p_i be given by (1.4) with probability $\frac{1}{2}$ and its translate with probability $\frac{1}{2}$, one obtains a stationary, ergodic sequence with nonreversible stationary distributions. So, in order to conclude that all stationary distributions are reversible, it is not enough to assume that $p_i > \frac{1}{2}$ and $p_i < \frac{1}{2}$, each a positive proportion of the time.

2 Sufficient Conditions for All Stationary Distributions to be Reversible

In this section, we prove Theorem 1 and Proposition 1.

Proof of Theorem 1 The proof is based on the proofs of Theorems 1.2 and 1.3 of [6], so we will only sketch the parts that are similar. We will use the basic coupling of two copies η_t and η'_t of the process. In this coupling, particles move together as much as possible. At rate p_i, for example, $(\eta,\eta') \to$

$$\begin{cases} (\eta_{i,i+1},\eta') & \text{if } \eta(i)=1, \eta(i+1)=0;\ \eta'(i)=\eta'(i+1) \text{ or } \eta'(i)=0, \eta'(i+1)=1,\\ (\eta,\eta'_{i,i+1}) & \text{if } \eta'(i)=1, \eta'(i+1)=0;\ \eta(i)=\eta(i+1) \text{ or } \eta(i)=0, \eta(i+1)=1,\\ (\eta_{i,i+1},\eta'_{i,i+1}) & \text{if } \eta(i)=\eta'(i)=1;\ \eta(i+1)=\eta'(i+1)=0. \end{cases}$$

Quantities related to the coupled process will be denoted by tildes. For example, the set of stationary distributions for the coupled process will be called $\tilde{\mathcal{I}}$.

For $m<n$, define the following two functions of a coupled configuration:

$$f_{m,n}(\eta,\eta') = \sum_{k=m}^{n} |\eta(k)-\eta'(k)|,$$

$$g_{m,n}(\eta,\eta') = \#\text{ of strict sign changes in the sequence } \{\eta'(m)-\eta(m),\dots,\eta'(n)-\eta(n)\}.$$

The fundamental property of these functions that makes them useful is that they cannot increase except as the result of transitions across the boundaries of $[m,n]$. Interior transitions can make them decrease unless $\eta_t \le \eta'_t$ or $\eta_t \ge \eta'_t$ in the case of $f_{m,n}$, and unless there is at most one sign change in the case of $g_{m,n}$.

We will use the following notation. If ν is a probability measure on $\{0,1\}^{Z^1}\times\{0,1\}^{Z^1}$, then $\nu\{(\eta,\eta'):\eta(i)=\delta,\eta'(i)=\delta'\}$ will be denoted by

$$\nu\begin{pmatrix}\delta'\\ \delta\\ i\end{pmatrix},$$

with analogous notation for probabilities of cylinder sets involving more than one site.

If $\nu\in\tilde{\mathcal{I}}$, then $\int \tilde{\mathcal{L}} f_{m,n}\, d\nu = 0$. Writing this out gives

$$\begin{aligned} 2\sum_{i=m}^{n-1}(p_i+q_{i+1})&\left[\nu\begin{pmatrix}1 & 0\\ 0 & 1\\ i & i+1\end{pmatrix}+\nu\begin{pmatrix}0 & 1\\ 1 & 0\\ i & i+1\end{pmatrix}\right]\\ &= p_{m-1}\left[\nu\begin{pmatrix}1 & 0\\ 0 & 0\\ m-1 & m\end{pmatrix}-\nu\begin{pmatrix}1 & 1\\ 1 & 0\\ m-1 & m\end{pmatrix}+\nu\begin{pmatrix}0 & 0\\ 1 & 0\\ m-1 & m\end{pmatrix}-\nu\begin{pmatrix}1 & 0\\ 1 & 1\\ m-1 & m\end{pmatrix}\right]\\ &\quad + q_m\left[\nu\begin{pmatrix}1 & 1\\ 0 & 1\\ m-1 & m\end{pmatrix}-\nu\begin{pmatrix}0 & 1\\ 0 & 0\\ m-1 & m\end{pmatrix}+\nu\begin{pmatrix}0 & 1\\ 1 & 1\\ m-1 & m\end{pmatrix}-\nu\begin{pmatrix}0 & 0\\ 0 & 1\\ m-1 & m\end{pmatrix}\right]\\ &\quad + \text{similar terms coming from the right boundary of } [m,n]. \end{aligned} \tag{2.1}$$

The left side of (2.1) represents interior transitions that lead to the loss of (two) discrepancies, while the right side corresponds to the gain or loss of discrepancies in $[m,n]$ due to transitions across the boundary.

At this point, the argument differs somewhat according to whether one, both, or neither of the following hold:

$$\inf_{i<0}(p_i+q_{i+1})>0, \qquad \inf_{i>0}(p_i+q_{i+1})>0. \tag{2.2}$$

If both of these infima are 0, then one can let $m\to-\infty$, $n\to+\infty$ in (2.1) along appropriate subsequences to conclude that

$$\sum_{i=-\infty}^{\infty}(p_i+q_{i+1})\left[\nu\begin{pmatrix}1 & 0\\ 0 & 1\\ i & i+1\end{pmatrix}+\nu\begin{pmatrix}0 & 1\\ 1 & 0\\ i & i+1\end{pmatrix}\right]=0. \tag{2.3}$$

It follows from this that ν puts no mass on configurations with adjacent discrepancies of opposite types, and then using the invariance of ν again, that it puts no mass on configurations with discrepancies of opposite type at all. Therefore, it follows that

$$\nu\{(\eta,\eta'):\eta\le\eta' \text{ or } \eta'\le\eta\}=1. \tag{2.4}$$

We will now assume that (2.2) holds. (The argument in the case that one of the infima in (2.2) is zero and the other is positive is a combination of these two arguments, and will be omitted.) It now follows that

$$\sum_{i=-\infty}^{\infty}\left[\nu\begin{pmatrix}1 & 0\\ 0 & 1\\ i & i+1\end{pmatrix}+\nu\begin{pmatrix}0 & 1\\ 1 & 0\\ i & i+1\end{pmatrix}\right]<\infty.$$

Using the invariance of ν again, it follows that

$$\sum_{i=-\infty}^{\infty}\left[\nu\begin{pmatrix}1 & 0\\ 0 & 1\\ i & i+k\end{pmatrix}+\nu\begin{pmatrix}0 & 1\\ 1 & 0\\ i & i+k\end{pmatrix}\right]<\infty$$

for any $k\ge 1$. (See the proof of Lemma 4.4 of [6] for details.) Next, using the fact that $\nu\in\tilde{\mathcal{I}}$ implies $\int\tilde{\mathcal{L}}g_{m,n}d\nu=0$, one can show that ν concentrates on configurations satisfying $g_{m,n}(\eta,\eta')\le 1$ for all $m<n$. (See the proofs of Lemmas 4.7 and 4.8 of [6].)

The conclusion is the following: In all cases, ν concentrates on configurations (η,η') with the property that there is at most one strict sign change in the doubly infinite sequence

$$\{\ldots,\eta'(-2)-\eta(-2),\eta'(-1)-\eta(-1),\eta'(0)-\eta(0),\eta'(1)-\eta(1),\eta'(2)-\eta(2),\ldots\},$$

i.e., such that exactly one of the following is true:

(a) $\eta=\eta'$.
(b) $\eta\le\eta'$ and $\eta\ne\eta'$.
(c) $\eta\ge\eta'$ and $\eta\ne\eta'$.
(d) There is a k so that $\eta(i)\le\eta'(i)$ for all $i\le k$ with infinitely many strict inequalities, and $\eta(i)\ge\eta'(i)$ for all $i>k$ with infinitely many strict inequalities.
(e) There is a k so that $\eta(i)\ge\eta'(i)$ for all $i\le k$ with infinitely many strict inequalities, and $\eta(i)\le\eta'(i)$ for all $i>k$ with infinitely many strict inequalities.

(The fact that finitely many strict inequalities is excluded in cases (d) and (e) is a consequence of the fact that the system is in equilibrium; if there were finitely many, there would be some rate at which they would be annihilated, and since they cannot be created, this would contradict stationarity.) Since each of the above sets of configurations is closed for the evolution of the coupled process, if $\nu \in \tilde{\mathcal{I}}_e$, then ν concentrates on exactly one of these sets.

Now take $\mu, \mu' \in \mathcal{I}_e$ such that $\phi(\mu) = \phi(\mu') = 0$. By Proposition 2.14 of Chap. VIII of [9], there is a $\nu \in \tilde{\mathcal{I}}_e$ with marginals μ and μ' respectively. We will show next that ν cannot concentrate on either of the last two sets described above. Suppose, for example, that it concentrates on the set described in (d). The k appearing there is random, so we will denote it by $K(\eta, \eta')$. Let

$$u_m = p_{m-1}\mu\{\eta : \eta(m-1) = 1, \eta(m) = 0\} = q_m\mu\{\eta : \eta(m-1) = 0, \eta(m) = 1\},$$
$$u'_m = p_{m-1}\mu'\{\eta : \eta(m-1) = 1, \eta(m) = 0\} = q_m\mu'\{\eta : \eta(m-1) = 0, \eta(m) = 1\}.$$

The second equality in each case comes from the fact that the fluxes are zero. Then

$$\begin{aligned}
\frac{u_m - u'_m}{p_{m-1}} &= \mu\{\eta : \eta(m-1) = 1, \eta(m) = 0\} - \mu'\{\eta : \eta(m-1) = 1, \eta(m) = 0\} \\
&= \nu\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ m-1 & m \end{pmatrix} + \nu\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ m-1 & m \end{pmatrix} + \nu\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ m-1 & m \end{pmatrix} \\
&\quad - \nu\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ m-1 & m \end{pmatrix} - \nu\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ m-1 & m \end{pmatrix} - \nu\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ m-1 & m \end{pmatrix}.
\end{aligned}$$

The third term on the right above is zero since ν concentrates on the set described in (d). The second, fourth and sixth terms are bounded by

$$\nu\{(\eta, \eta') : K(\eta, \eta') \le m\}. \tag{2.5}$$

For the sixth term, for example, note that $\eta'(m-1) = 1, \eta'(m) = 0, \eta(m-1) = 0, \eta(m) = 1$ implies that $K(\eta, \eta') = m - 1$. The probability in (2.5) tends to zero as $m \to -\infty$. Therefore,

$$p_{m-1}\left[\nu\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ m-1 & m \end{pmatrix} - \nu\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ m-1 & m \end{pmatrix}\right] - (u_m - u'_m) \to 0$$

as $m \to -\infty$. Similarly

$$q_m\left[\nu\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ m-1 & m \end{pmatrix} - \nu\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ m-1 & m \end{pmatrix}\right] - (u_m - u'_m) \to 0$$

as $m \to -\infty$. Taking differences, we see that

$$p_{m-1}\left[\nu\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ m-1 & m \end{pmatrix} - \nu\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ m-1 & m \end{pmatrix}\right] - q_m\left[\nu\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ m-1 & m \end{pmatrix} - \nu\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ m-1 & m \end{pmatrix}\right]$$

tends to zero as $m \to -\infty$. Note that this says that the sum of the first, second, fifth and sixth terms on the right side of (2.1) tends to zero as $m \to -\infty$. The third, fourth, seventh and eighth terms are bounded by (2.5), so they tend to zero individually as $m \to -\infty$. Applying a similar argument to the terms in (2.1) that come from the right boundary of $[m, n]$, we see that the entire right side of (2.1) tends to zero as $m \to -\infty$, $n \to \infty$. It follows by passing to the limit in (2.1) that (2.3) holds, and then that (2.4) holds, contradicting the assumption that ν concentrates on the set described in (d). A similar argument shows that (e) cannot hold either. Therefore, either $\mu \leq \mu'$ or $\mu' \leq \mu$.

To complete the proof of Theorem 1, take $\mu \in \mathcal{I}_e$ and let μ' be one of the extremal reversible measures—ν_α or μ_n according to whether (1.2) holds or not. For each such choice of μ', we now know that either $\mu \leq \mu'$ or $\mu' \leq \mu$. Suppose first that (1.2) holds. Then there is an $\alpha^* \in [0, 1]$ so that

$$\mu \quad \begin{cases} \leq \nu_\alpha & \text{if } \alpha \geq \alpha^*, \\ \geq \nu_\alpha & \text{if } \alpha \leq \alpha^*. \end{cases}$$

Since the ν_α's, together with the pointmasses on the configurations that are $\equiv 0$ and $\equiv 1$, form a weakly continuous one parameter family, it follows that $\mu = \nu_{\alpha^*}$ if $\alpha^* \in (0, 1)$, and is the pointmass on the $\equiv 0$ or $\equiv 1$ configuration if $\alpha^* = 0$ or 1.

If (1.2) fails, then it follows that μ concentrates on $\cup_n A_n \cup \{0, 1\}$, where 0 and 1 represent the identically 0 and identically 1 configurations respectively, so that the statement of the theorem holds in this case as well. □

Proof of Proposition 1 We will prove the first statement; the proof of the second is similar. Suppose that $p_i \geq \frac{1}{2} - \epsilon$ for $k \leq i \leq k + n$, and take $\mu \in \mathcal{I}$. Summing (1.3) gives

$$\begin{aligned} n\phi(\mu) &= \sum_{i=k}^{k+n-1} [p_i \mu\{\eta : \eta(i) = 1, \eta(i+1) = 0\} - q_{i+1}\mu\{\eta : \eta(i) = 0, \eta(i+1) = 1\}] \\ &\geq \sum_{i=k}^{k+n-1} \left[\left(\frac{1}{2} - \epsilon\right) \mu\{\eta : \eta(i) = 1, \eta(i+1) = 0\} \right. \\ &\quad \left. - \left(\frac{1}{2} + \epsilon\right) \mu\{\eta : \eta(i) = 0, \eta(i+1) = 1\} \right] \\ &= -2\epsilon \sum_{i=k}^{k+n-1} \mu\{\eta : \eta(i) = 1, \eta(i+1) = 0\} \\ &\quad + \left(\frac{1}{2} + \epsilon\right) \sum_{i=k}^{k+n-1} [\mu\{\eta : \eta(i) = 1\} - \mu\{\eta : \eta(i+1) = 1\}] \\ &\geq -2\epsilon n + \left(\frac{1}{2} + \epsilon\right) [\mu\{\eta : \eta(k) = 1\} - \mu\{\eta : \eta(k+n) = 1\}]. \end{aligned}$$

Since n is arbitrary, it follows that $\phi(\mu) \geq -2\epsilon$. To remove the extra factor of two, it suffices to note that

$$\sum_{i=k}^{k+n-1} \mu\{\eta : \eta(i) = 1, \eta(i+1) = 0\} \leq \frac{n}{2}$$

for even n, since the events occuring in consecutive summands above are disjoint. □

3 Sufficient Conditions for the Existence of Nonreversible Stationary Distributions

In this section, we prove Theorem 2 and Corollary 2.

Proof of Theorem 2 Consider two choices $p_{m-1}, \dots, p_n$ and $p'_{m-1}, \dots, p'_n$ of jump probabilities satisfying $p'_i \geq p_i$ for each i. Quantities related to the process corresponding to the p'_i's will be identified with a prime. We need to show that $\phi(\mu'_{m,n}) \geq \phi(\mu_{m,n})$. In order to do so, we construct a coupled process as follows: $(X_t, \eta_t, \eta'_t, Y_t)$ has state space

$$\left\{(x,\eta,\eta',y) \in Z^1 \times \{0,1\}^{[m,n]} \times \{0,1\}^{[m,n]} \times Z^1 : x + \sum_{i=m}^{n}[\eta(i) - \eta'(i)] + y = 0 \right.$$
$$\left. \text{and } x + \sum_{i=m}^{k}[\eta(i) - \eta'(i)] \geq 0 \text{ for all } m-1 \leq k \leq n \right\}.$$

Transitions inside $[m,n]$ correspond to letting particles in the two configurations move together as much as possible. Transitions across the boundaries $(m-1,m)$ and $(n,n+1)$ follow the same rules, with X_t and Y_t keeping track of the number of discrepancies that leave or enter $[m,n]$ at the left or right, respectively. To be more explicit, if the 1 0 appearing below are at sites $i, i+1$ respectively (with $m \leq i < n$), then

$$\begin{pmatrix}\eta'\\ \eta\end{pmatrix} = \begin{pmatrix}\cdots & 1 & 0 & \cdots\\ & 1 & 0 & \end{pmatrix} \to \begin{cases} \begin{pmatrix}\cdots & 0 & 1 & \cdots\\ & 1 & 0 & \end{pmatrix} & \text{at rate } p'_i - p_i, \\ \begin{pmatrix}\cdots & 0 & 1 & \cdots\\ & 0 & 1 & \end{pmatrix} & \text{at rate } p_i, \end{cases}$$

while if the 0 1 appearing below are at sites $i, i+1$ respectively, for example, then

$$\begin{pmatrix}\eta'\\ \eta\end{pmatrix} = \begin{pmatrix}\cdots & 0 & 1 & \cdots\\ & 0 & 1 & \end{pmatrix} \to \begin{cases} \begin{pmatrix}\cdots & 0 & 1 & \cdots\\ & 1 & 0 & \end{pmatrix} & \text{at rate } q_{i+1} - q'_{i+1}, \\ \begin{pmatrix}\cdots & 1 & 0 & \cdots\\ & 1 & 0 & \end{pmatrix} & \text{at rate } q'_{i+1}. \end{cases}$$

Note that in both cases, the discrepancy $\binom{0}{1}$ is produced to the left of the discrepancy $\binom{1}{0}$. This means that the inequality $x + \sum_{i=m}^{k}[\eta(i) - \eta'(i)] \geq 0$ is not violated by these transitions.

At the left boundary, one has, for example, the following transitions:

$$\left(x, \begin{pmatrix}\eta'\\ \eta\end{pmatrix}\right) = \left(x, \begin{pmatrix}0 & \cdots\\ 0 & \end{pmatrix}\right) \to \begin{cases} \left(x+1, \begin{pmatrix}1 & \cdots\\ 0 & \end{pmatrix}\right) & \text{at rate } p'_{m-1} - p_{m-1}, \\ \left(x, \begin{pmatrix}1 & \cdots\\ 1 & \end{pmatrix}\right) & \text{at rate } p_{m-1}, \end{cases}$$

or

$$\left(x, \begin{pmatrix}\eta'\\ \eta\end{pmatrix}\right) = \left(x, \begin{pmatrix}1 & \cdots\\ 0 & \end{pmatrix}\right) \to \left(x-1, \begin{pmatrix}1 & \cdots\\ 1 & \end{pmatrix}\right) \quad \text{at rate } p_{m-1}.$$

While we have not listed all the possible transitions, hopefully we have listed enough so that the reader can easily construct the others.

Now, start this process at $(0,\eta,\eta,0)$, where η is any point in $\{0,1\}^{[m,n]}$. The limiting distribution as $t\to\infty$ of η_t is $\mu_{m,n}$ while the limiting distribution of η'_t is $\mu'_{m,n}$. For fixed $m\le k<n$, let

$$\begin{aligned} N_t = {} & (\text{the number of times a particle in } \eta_t \text{ has crossed from } k \text{ to } k+1 \text{ by time } t) \\ & - (\text{the number of times a particle in } \eta_t \text{ has crossed from } k+1 \text{ to } k \text{ by time } t), \end{aligned}$$

with N'_t being defined in an analogous way in terms of the process η'_t. Then one can easily check that

$$N'_t - N_t = X_t + \sum_{i=m}^{k}[\eta_t(i) - \eta'_t(i)],$$

so that $N'_t \ge N_t$ a.s. On the other hand,

$$\frac{d}{dt}EN_t = p_k P^\eta[\eta_t(k)=1,\eta_t(k+1)=0] - q_{k+1}P^\eta[\eta_t(k)=0,\eta_t(k+1)=1],$$

so that

$$\phi(\mu_{m,n}) = \lim_{t\to\infty}\frac{d}{dt}EN_t = \lim_{t\to\infty}\frac{EN_t}{t}.$$

It follows from these observations that $\phi(\mu'_{m,n})\ge\phi(\mu_{m,n})$ as required. □

Proof of Corollary 2 Let $\mu_{m,n}$ be the stationary measure for the process on $[m,n]$ corresponding to the given p_i's, and $\mu'_{m,n}$ be the one for the process with $p'_i\equiv\frac12+\epsilon$. By Theorem 2, $\phi(\mu_{m,n})\ge\phi(\mu'_{m,n})$. By Theorem 2.9 of [7],

$$\lim_{\substack{m\to-\infty\\ n\to+\infty}} \mu'_{m,n} = \nu_{1/2},$$

and therefore,

$$\lim_{\substack{m\to-\infty\\ n\to+\infty}} \phi(\mu'_{m,n}) = \frac{\epsilon}{2}.$$

It follows that any weak limit μ of $\mu_{m,n}$ as $m\to-\infty, n\to+\infty$ is in $\mathcal{I}$ and satisfies $\phi(\mu)\ge\frac{\epsilon}{2}$. The measure μ is not reversible since all reversible measures have zero flux. □

4 The Noninteracting Case

In this section, we consider a system of independent particles, each of which evolves as a continuous time Markov chain on Z^1 with unit exponential holding times and transition probabilities p_i and q_i from i to $i+1$ and $i-1$ respectively. It has been known since at least the publication of Doob's classic 1953 book [3] that one way to construct stationary distributions for this system is to let $\{\eta(i), i\in Z^1\}$ be independent Poisson random variables with $E\eta(i)=\sigma_i$, where $\sigma=\{\sigma_i\}$ is an invariant measure for the corresponding one-particle motion:

$$\sigma_i = \sigma_{i-1}p_{i-1} + \sigma_{i+1}q_{i+1}. \tag{4.1}$$

By Theorem 4.12 of [8], these provide all of the extremal stationary distributions if the one-particle chain is not positive recurrent.

In such a stationary distribution, the flux of particles between i and $i+1$ is

$$\phi = p_i \sigma_i - q_{i+1}\sigma_{i+1}, \tag{4.2}$$

which is independent of i by (4.1).

Solving (4.2) recursively leads to

$$\sigma_n = \frac{p_{n-1}\cdots p_m}{q_n \cdots q_{m+1}}\sigma_m - \frac{\phi}{q_n}\left(1 + \frac{p_{n-1}}{q_{n-1}} + \cdots + \frac{p_{n-1}\cdots p_{m+1}}{q_{n-1}\cdots q_{m+1}}\right)$$

for $m < n$. It follows that if $\phi > 0$, there is a positive solution σ to (4.2) if and only if

$$1 + \frac{q_0}{p_0} + \frac{q_0 q_1}{p_0 p_1} + \cdots < \infty,$$

while if $\phi < 0$, there is a positive solution to (4.2) if and only if

$$1 + \frac{p_0}{q_0} + \frac{p_0 p_{-1}}{q_0 q_{-1}} + \cdots < \infty.$$

We therefore have the following result:

Theorem 4 *Suppose the p_i's are i.i.d. Then there is an extremal stationary distribution for the independent particle system with positive flux if $E\log(p_0/q_0) > 0$, and one with negative flux if $E\log(q_0/p_0) > 0$.*

Not surprisingly, these are exactly the conditions for a random walk in a random environment to be transient to the right or left respectively—see Theorem 1.7 in [10].

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